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TOTAL DIFFERENTIABILITY. A CORRECTION.

BY E. J. TOWNSEND.

In my paper on Total Differentiability in the September number of the Annals the theorem is incorrect as stated. The boundedness condition should be placed on the second derivative with respect to x or to y and not on the first. However, one may replace this boundedness condition by the more restricted condition of the uniform convergence of either of the difference quotients

$$\frac{f(x_0 + \Delta x, y) - f(x_0, y)}{\Delta x}, \qquad \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y}.$$

If we assume the uniform convergence of the first of these difference quotients, the theorem may be stated as follows:

THEOREM. *Given a function $f(x, y)$ having at the point (x_0, y_0) the partial derivatives f_x', f_y'; then $f(x, y)$ is totally differentiable at (x_0, y_0) if f_x' is continuous in y at (x_0, y_0) and in the neighborhood of this point the difference quotient*

$$\frac{f(x_0 + \Delta x, y) - f(x_0, y)}{\Delta x} \tag{a}$$

converges uniformly as to Δx.

We have the identity

$$\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta(x, y)} - \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta(x, y)} - \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta(x, y)} = 0, \tag{1}$$

where $\Delta(x, y) \equiv \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Since the partial derivative $f_y'(x_0, y_0)$ exists, we may write

$$\left| \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} - f_y'(x_0, y_0) \right| < \epsilon,$$

or

$$|f(x_0, y_0 + \Delta y) - f(x_0, y_0) - \Delta y f_y'(x_0, y_0)| < \epsilon |\Delta y|, \qquad |\Delta y| < \delta_1. \tag{2}$$

From the convergence of (a) it follows that f_x' exists for all values of y in the neighborhood of (x_0, y_0) and by hypothesis it is continuous in y at this point. We then have

$$|f_x'(x_0, y_0 + \Delta y) - f_x'(x_0, y_0)| < \frac{\epsilon}{2}, \qquad |\Delta y| < \delta_2. \tag{3}$$

From the uniform convergence of the difference quotient (*a*) as to Δx for all values of y in the neighborhood of (x_0, y_0), we have

$$\left|\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} - f_x'(x_0, y_0 + \Delta y)\right| < \frac{\epsilon}{2}, \qquad |\Delta x| < \lambda, \quad |\Delta y| < \delta_3, \tag{4}$$

where λ is independent of the choice of Δy so long as Δy is less than δ_3 in absolute value. Combining the inequalities (3) and (4) we have

$$\left|\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} - f_x'(x_0, y_0)\right| < \epsilon,$$

$$|\Delta x| < \lambda, \qquad |\Delta y| < \delta_4,$$

where δ_4 is the smaller of the two numbers δ_2, δ_3. We then have

$$|f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) - \Delta x f_x'(x_0, y_0)| < \epsilon \cdot |\Delta x|, \qquad |\Delta x| < \lambda, \quad |\Delta y| < \delta_4. \tag{5}$$

Making use of (2) and (5) we have from (1) for $|\Delta x| < \lambda$, $|\Delta y| < \delta$, δ being the smaller of the two numbers δ_1 and δ_4,

$$\left|\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - \Delta x f_x'(x_0, y_0) - \Delta y f_y'(x_0, y_0)}{\Delta(x, y)}\right| < \epsilon \cdot \frac{|\Delta x| + |\Delta y|}{\Delta(x, y)} \leqq \epsilon\sqrt{2}.$$

Hence the first member of this inequality has the limit zero as Δx, Δy approach zero simultaneously, and $f(x, y)$ is totally differentiable at (x_0, y_0) as the theorem states.

A corresponding statement of the theorem may be made involving the continuity of the partial derivative f_y' with respect x at the point (x_0, y_0) and the uniform convergence of the difference quotient

$$\frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y}$$

in the neighborhood of that point. In the theorem one may replace the continuity of f_x' with respect to y by the continuity of the given function $f(x, y)$ with respect to y for all values of x in the neighborhood (x_0, y_0), since then by virtue of the uniform convergence of the difference quotient (*a*) the partial derivative f_x' is continuous in y at (x_0, y_0). It follows from this theorem that $f(x, y)$ is totally differentiable at any point (x_0, y_0) of a given region R if f_x', f_y' exist, are continuous in x alone and in y alone and either converges uniformly to its values in the neighborhood of that point.